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Application of weak equivalence transformations to a group analysis of a drift–diffusion model

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Abstract. A group analysis of a class of drift–diffusion systems is performed. In account of the presence of arbitrary constitutive functions, we look for Lie symmetries starting from the *weak equivalence transformations*. Applications to the transport of charges in semiconductors are presented and a special class of solutions is given for particular doping profiles.

1. Introduction

In this paper we tackle the problem of finding symmetries of the class $\mathcal{S}$ of drift–diffusion systems of partial differential equations (PDEs), which arise in several problems of physical (e.g. semiconductors) or biological (e.g. evolution of tumours) interest

$$u^1_{x^1} = [p^1 u^1_{x^2} - p^2 u^1]_{x_2} \tag{1}$$

$$u^2_{x^2} = u^1 - p^3 \tag{2}$$

where

$$p^1 = p^1(u^1) \qquad p^2 = p^2(u^2) \qquad p^3 = p^3(x^2) \tag{3}$$

are sufficiently smooth arbitrary functions of their arguments. A special subclass of this system was previously considered by the authors in [1].

The search for symmetries is usually performed via the direct Lie approach [2–6]. This procedure requires the solution of the so-called *determining* system, which is an overdetermined system of PDEs, linear in the components of the infinitesimal generators. However, since in (1), (2), the functions, not *a priori* assigned, p^1, p^2 and p^3 appear, the computational difficulties considerably increase and it is too involved to get the complete symmetry classification by the Lie direct method. Therefore, it is convenient to proceed to look for *equivalence transformations* or *weak equivalence transformations* as in [7–12] where the problem of symmetry classification in the presence of arbitrary functions has been considered for different physical models.

Roughly speaking, an equivalence transformation (see section 2 for a more formal definition) is a change of variables which transforms the original system of PDEs into a new system having the same differential structure, but in which the transformed arbitrary functions might have a different form even if they continue to depend on the same arguments.

Instead, a weak equivalence transformation (WET) can also change the arguments of the transformed functions: e.g., $p^1(u^1)$ can be transformed into $\overline{p^1}(\overline{x}^i, \overline{u}^j)$.

The use of equivalence transformations for finding symmetries [7, 8] is based on the fact that, from their infinitesimal generators, one gets, under suitable hypothesis, the infinitesimal generators of the Lie symmetries by projection. The same strategy is followed when WETs are employed [9–12] but this latter procedure, as shown in [1, 9–12], usually gives a wider set of symmetries. A procedure based on equivalence or WETs does not, in general, ensure determination of the complete symmetry classification, but in applications it reveals a successful and computationally appealing way to obtain symmetries.

With respect to the aforesaid papers, we suggest an improved method for finding WETs. Following [13, 14], we shall include the dependence on the arbitrary elements in all the components of the generator by generalizing the method proposed in [2].

As an application of the results of the classification we shall get reduced systems for the drift–diffusion model (DDM) of semiconductors and a class of exact solutions will also be presented.

The plan of the paper is as follows. In section 2 we recall the concept of weak equivalence transformation and illustrate an infinitesimal method to obtain them. In section 3 we look for weak equivalence transformations for the class of drift–diffusion systems under consideration. In section 4 Lie symmetries are obtained by projection of the WETs and, in section 5, some applications to the DDM of semiconductors are presented. Finally, in the appendix, the general formulae for the prolongations of arbitrary order of the infinitesimal generators of equivalence transformations are obtained.

2. On the equivalence transformations and their infinitesimal generators

We recall that a continuous equivalence transformation (CET) [2] for the system (1), (2) is a transformation of the type†

$$x^i = x^i(\overline{x}^j, \overline{u}^\beta, \overline{p}^B) \tag{4}$$

$$u^\alpha = u^\alpha(\overline{x}^j, \overline{u}^\beta, \overline{p}^B) \tag{5}$$

$$p^A = p^A(\overline{x}^j, \overline{u}^\beta, \overline{p}^B) \tag{6}$$

which is locally a C^∞-diffeomorphism and changes the original system into a new system having the same differential structure but a different form of the arbitrary functions p^A, that is, in general,

$$\overline{p}^1(\overline{u}^1) \neq p^1(\overline{u}^1) \qquad \overline{p}^2(\overline{u}^2) \neq p^2(\overline{u}^2) \qquad \overline{p}^3(\overline{x}^2) \neq p^3(\overline{x}^2).$$

A continuous invariant transformation can be regarded as a particular CET such that

$$\overline{p}^1(\overline{u}^1) = p^1(\overline{u}^1) \qquad \overline{p}^2(\overline{u}^2) = p^2(\overline{u}^2) \qquad \overline{p}^3(\overline{x}^2) = p^3(\overline{x}^2).$$

Therefore, the continuous invariant transformations are a subset of the set of the equivalence transformations.

The direct search for the equivalence transformations through the finite form of the transformation encounters considerable computational difficulties. A way to overcome these problems was indicated by Ovsiannikov [2] who suggested using the Lie infinitesimal criterion, giving an algorithm to find the infinitesimal generators of the CETs. The original method of [2]

† The Latin indices run from one to the number of independent variables ($i = 1, 2$), the Greek indices run from one to the number of dependent variables ($\beta = 1, 2$) and the capital Latin letters run from one to the number of arbitrary functions ($A = 1, 2, 3$).

was employed in several papers (see [15,16] and references therein). Recently, a generalization has been presented in [13, 14].

In the *basic augmented space* $A = X \times U \times P$, where $\{(x^1, x^2)\} = X \subseteq \mathcal{R}^2$, $\{(u^1, u^2)\} = U \subseteq \mathcal{R}^2$ and $\{(p^1, p^2, p^3)\} = P \subseteq \mathcal{R}^3$, let us consider a one-parameter group of transformations

$$x^i = x^i(\overline{x}^j, \overline{u}^\beta, \overline{p}^B, \varepsilon) \tag{7}$$

$$u^\alpha = u^\alpha(\overline{x}^j, \overline{u}^\beta, \overline{p}^B, \varepsilon) \tag{8}$$

$$p^A = p^A(\overline{x}^j, \overline{u}^\beta, \overline{p}^B, \varepsilon) \tag{9}$$

which is locally a C^∞-diffeomorphism, depends analytically on the parameter ε in a neighbourhood of $\varepsilon = 0$ and reduces to the identity transformation for $\varepsilon = 0$.

The infinitesimal generator of the transformation (7)–(9) has the form

$$\Gamma = \xi^i \partial_{x_i} + \eta^\alpha \partial_{u^\alpha} + \mu^A \partial_{p^A}. \tag{10}$$

Following [13], we consider both ξ^i, η^α and μ^A as functions of x^i, u^α and p^A. In the previous papers the same procedure was followed but the dependence on p^A was assumed only for μ^A while the components ξ^i and η^α of Γ were sought as functions of x^i and u^α alone according to [2].

In order to make the notation as compact as possible, we put

$$z^a := (x^i, u^\alpha) \qquad \text{and} \qquad \nu^a := (\xi^i, \eta^\alpha)$$

and write Γ as

$$\Gamma = \nu^a \partial_{z^a} + \mu^A \partial_{p^A}.$$

Since the systems belonging to the class $\mathcal{S}$ involve second-order derivatives, we need the first and second prolongation $\Gamma^{(1)}$ and $\Gamma^{(2)}$ of Γ.

The general expression of $\Gamma^{(1)}$ has been given in [13]. For the transformation (7)–(9) in the *first-order jet space* [3]

$$A^{(1)} = A \times A_1 = A \times \{(u^\alpha_{x^j}, p^A_{x^j}, p^B_{u_k})\}$$

we have

$$\Gamma^{(1)} = \Gamma + \zeta^\alpha_j \partial_{u^\alpha_j} + \omega^A_a \partial_{p^A_a} \tag{11}$$

where

$$u^\alpha_i = u^\alpha_{x^i} \qquad \text{and} \qquad p^A_a = p^A_{z^a}$$

$$\zeta^\alpha_j = D^e_j \eta^\alpha - u^\alpha_k D^e_j \xi^k \tag{12}$$

$$\omega^A_a = \tilde{D}^e_a \mu^A - p^A_b \tilde{D}^e_a \nu^b \tag{13}$$

with

$$D^e_j = \partial_{x^j} + u^\alpha_j \partial_{u^\alpha} + (p^A_{u^\alpha} u^\alpha_j + p^A_{x^j}) \partial_{p^A} \tag{14}$$

$$\tilde{D}^e_a = \partial_{z^a} + p^A_a \partial_{p^A}. \tag{15}$$

We have generalized this result to prolongations of Γ of arbitrary order. For the sake of clarity we report such a generalization in the appendix. Here we only need the prolongation

$$\Gamma^* = \Gamma^{(1)} + \zeta^\alpha_{ij} \partial_{u^\alpha_{ij}} \tag{16}$$

where

$$\zeta^\alpha_{ij} = D^e_j \zeta^\alpha_i - u^\alpha_{ik} D^e_j \xi^k$$

where Γ^* is the specialization of $\Gamma^{(2)}$ (see the appendix) to our case.

In order to take into account the functional dependences (3) in searching for the equivalence transformation, as indicated in [2], we have to impose the invariance of the system (1), (2) and, moreover, the invariance of the additional relations

$$p^1_{x^i} = 0 \qquad p^1_{u^2} = 0 \qquad p^2_{x^i} = 0 \qquad p^2_{u^1} = 0 \qquad p^3_{u^\alpha} = 0 \qquad p^3_{x^1} = 0 \tag{17}$$

under the action of the generator Γ^*, that is

$$\Gamma^*(u^1_{x^1} - [p^1 u^1_{x^2} - p^2 u^1]_{x_2})|_{(1);(2);(17)} = 0 \tag{18}$$

$$\Gamma^*(u^2_{x^2} - u^1 + p^3)|_{(1);(2);(17)} = 0 \tag{19}$$

$$(\Gamma^* p^1_{x^i})|_{(1);(2);(17)} = 0 \qquad (\Gamma^* p^1_{u^2})|_{(1);(2);(17)} = 0, \tag{20}$$

$$(\Gamma^* p^2_{x^i})|_{(1);(2);(17)} = 0 \qquad (\Gamma^* p^2_{u^1})|_{(1);(2);(17)} = 0 \tag{21}$$

$$(\Gamma^* p^3_{u^\alpha})|_{(1);(2);(17)} = 0 \qquad (\Gamma^* p^3_{x^1})|_{(1);(2);(17)} = 0 \tag{22}$$

where, as usual, the subscripts mean that the previous conditions must be evaluated by taking into account the relations (1), (2), (17).

If the invariance of the conditions (17) is not required, that is if only the conditions

$$\Gamma^*(u^1_{x^1} - [p^1 u^1_{x^2} - p^2 u^1]_{x_2})|_{(1);(2)} = 0 \tag{23}$$

$$\Gamma^*(u^2_{x^2} - u^1 + p^3)|_{(1);(2)} = 0 \tag{24}$$

are considered, one gets continuous transformations under which the transformed system maintains the same differential structure but the transformed p^A may depend on the other dependent and independent variables (e.g., $\overline{p}^1$ may also depend on $\overline{x}^i$ and $\overline{u}^2$).

We shall call the equivalence transformations obtained without imposing the additional conditions (20)–(22), WETs [12] for the system (1), (2) with the functional dependences (3).

3. Weak equivalence classification

As previously noticed, on account of the presence of the arbitrary functions p^A, to classify, following [2], the symmetry groups of the system (1), (2) is rather tedious even if symbolic manipulation packages are used, so we will follow a different approach which makes use of the WETs. First we look for a *weak equivalence classification*; that is, following [12], we classify in the augmented space $\tilde{A} = \{(x^1, x^2, u^1, u^2, p^2)\}$ the functional forms of $p^1 := \alpha(u^1)$ and $p^3 := \gamma(x^2)$ for which the system admits WETs. Then, by a suitable projection method [12], Lie symmetry infinitesimal generators are obtained from those of the WETs.

Therefore, we will seek the WETs of the system

$$u^1_{x^1} = (\alpha(u^1)u^2_{x^2})_{x^2} - u^1_{x^2} p - u^1 p_{u^2} u^2_{x^2} \tag{25}$$

$$u^2_{x^2} = u^1 - \gamma(x^2) \tag{26}$$

where the only arbitrary function p^2, denoted now by p, is assumed (because we are looking for WETs) to depend on x^j and $u^\alpha(x^k)$.

The generator in the appropriate second-order *jet space*

$$\tilde{A}^{(2)} = \tilde{A} \times \tilde{A}_1 \times \tilde{A}_2 = \tilde{A} \times \{(u^\alpha_{x^j}, p_{x^j}, p_{u^\beta})\} \times \{(u^\alpha_{x^i x^j}, p_{x^i x^j}, p_{u^\alpha x^j}, p_{u^\alpha u^\beta})\}$$

explicitly reads

$$\Gamma = \xi^i \partial_{x_i} + \eta^\alpha \partial_{u^\alpha} + \mu \partial_p \tag{27}$$

where the coordinates ξ^i, η^α and μ are sought depending on x^k, u^α and p.

Table 1. Weak equivalence classification. The infinitesimal generators appearing in the table are given by $\Gamma_1 = \partial_{x^1}$, $\Gamma_2 = \partial_{x^2} - Cu^1\partial_{u^1} - Cu^2\partial_{u^2}$, $\Gamma_3 = \frac{C}{k_1}x^1\partial_{x^1} + \partial_{x^2} - Cu^1\partial_{u^1} - Cu^2\partial_{u^2} - \frac{C}{k_1}p\partial_p$, $\Gamma_B = B(x^1)\partial_{u^2}$, $\Gamma_D = D(x^1)\partial_{x^2} + D'(x^1)\partial_p$, where $B(x^1)$ and $D(x^1)$ are arbitrary functions of x^1.

$\alpha(u^1)$	$\gamma(x^2)$	Infinitesimal generators
arbitrary	arbitrary	Γ_1, Γ_B
arbitrary	$\gamma_0 =$ constant	$\Gamma_1, \Gamma_B, \Gamma_D$
$\alpha_0 =$ constant	$\gamma_0 \exp(-Cx^2)$	$\Gamma_1, \Gamma_B, \Gamma_2$
$\tilde{k}(k_1u^1)^{1/k_1}, k_1 \neq 0$	$\gamma_0 \exp(-Cx^2)$	$\Gamma_1, \Gamma_B, \Gamma_3$

The invariance conditions

$$\Gamma^*(u^1_{x^1} - (\alpha(u^1)u^2_{x^2})_{x_2} + u^1_{x^2}p + u^1 p_{u^2}u^2_{x^2})|_{(25);(26)} = 0 \tag{28}$$

$$\Gamma^*(u^2 - u^1 + \gamma(x_2))|_{(25),(26)} = 0 \tag{29}$$

lead to the system

$$\xi^2_{x^2} - \eta^2_{u^1} = 0 \tag{30}$$

$$\eta^2_{u^2}(u^1 - \gamma(x^2)) + \eta^2_{x^2} - \eta^1 + \gamma'(x^2)\xi^2 = 0 \tag{31}$$

$$\alpha''(u^1)\eta^1 + \alpha'(u^1)\eta^1_{u^1} - \frac{(\alpha'(u^1))^2}{\alpha(u^1)}\eta^1 = 0 \tag{32}$$

$$\mu = 2\alpha(u^1)\eta^1_{u^1x^2} - p\xi^2_{x^2} + 2\alpha'(u^1)\eta^1_{x^2} + p\frac{\alpha'(u^1)}{\alpha(u^1)}\eta^1 + \xi^2_{x^1} - \alpha(u^1)\xi^2_{x^2x^2} \tag{33}$$

$$2\xi^2_{x^2} = \xi^1_{x^1} + \frac{\alpha'(u^1)}{\alpha(u^1)}\eta^1 \tag{34}$$

$$u^1\eta^2_{x^2} + u^1(\gamma(x^2) - u^1)\left(\frac{\alpha'(u^1)}{\alpha(u^1)}\eta^1 - 2\xi^2_{x^2} + \eta^1_{u^1} - \mu_p\right) + \eta^1(u^1 - \gamma(x^2)) = 0 \tag{35}$$

$$p\eta^1_{x^2} - u^1(\gamma(x^2) - u^1)\mu_p - \alpha(u^1)\eta^1_{x^2x^2} + \eta^1_{x^1} = 0 \tag{36}$$

for the functions

$$\xi^1 = \xi^1(x^1) \qquad \xi^2 = \xi^2(x^1, x^2) \qquad \eta^1 = \eta^1(x^1, x^2, u^1)$$
$$\eta^2 = \eta^2(x^1, x^2, u^1, u^2) \qquad \mu = \mu(x^1, x^2, u^1, u^2, p).$$

From the analysis of the previous system the classes of weak equivalence arise. We summarize them in table 1.

Note that a similar classification could also be performed with respect to only one of the p^A. In general, the choice of the functions to be classified depends on computational convenience or on physical considerations.

4. Lie symmetries via WETs

Starting from the weak equivalence classes found in the previous section, after observing that the components ξ^i and η^α of the infinitesimal WET generator do not depend on p, one can obtain Lie point symmetries of the system (1), (2) by using the procedure introduced in [12], based on the following theorem.

Theorem 1. *Let*

$$\Gamma = \xi^i(x^k, u^\beta)\partial_{x_i} + \eta^\alpha(x^k, u^\beta)\partial_{u^\alpha} + \mu(x^k, u^\beta, p)\partial_p$$

Table 2. Symmetries obtained via WET. The infinitesimal generators appearing in the table are defined as $X_1 = \partial_{x^1}$, $X_2 = \partial_{x^2}$, $X_3 = \frac{Cx^1}{k_1}\partial_{x^1} + \partial_{x^2} - Cu^1\partial_{u^1} - Cu^2\partial_{u^2}$, $X_4 = \partial_{x^2} - Cu^1\partial_{u^1} - Cu^2\partial_{u^2}$, $X_B = B(x^1)\partial_{u^2}$ and $X_D = D(x^1)\partial_{x^2} + \frac{D'(t)}{\phi_1}\partial_{u^2}$.

$\alpha(u^1)$	$\gamma(x^2)$	$\phi(u^2)$	Symmetry generators
arbitrary	arbitrary	arbitrary	X_1
arbitrary	arbitrary	ϕ_0	X_1, X_B
arbitrary	γ_0	arbitrary	X_1, X_2
arbitrary	γ_0	ϕ_0	X_1, X_2, X_B
arbitrary	γ_0	$\phi_1 u^2 + \phi_0, \phi_1 \neq 0$	X_1, X_D
α_0	$\gamma_0 \exp(-Cx^2)$	ϕ_0	X_1, X_B, X_3
$\tilde{k}(k_1 u^1)^{1/k_1}$	$\gamma_0 \exp(-Cx^2)$	$\phi_0(u^2 + r)^{1/k_1}$	X_1, X_3

be an infinitesimal generator of a WET for the system (1), (2). The projection of Γ

$$X = \xi^i(x^k, u^\beta)\partial_{x_i} + \eta^\alpha(x^k, u^\beta)\partial_{u^\alpha}$$

in the (x^i, u^α)*-space is an infinitesimal symmetry generator if and only if the specializations of the function* p *are invariant with respect to* Γ.

Thus, according to the above theorem, we require the invariance of the additional restriction

$$p = \phi(u^2) \tag{37}$$

under the action of the general infinitesimal generator Γ, which leads to

$$\Gamma(p - \phi(u^2))_{p=\phi(u^2)} = 0 \tag{38}$$

that is

$$\mu p - \phi'(u^2)\eta^2 = 0. \tag{39}$$

Specializing Γ for each equivalence class, we get the symmetry classification.

Let us consider, as an example, the class characterized by $\alpha(u^1)$ and $\gamma(x^2)$ arbitrary. Since, in this case (see table 1), $\Gamma = c_1\partial_{x_1} + B(x^1)\partial_{x^2}$, the condition (39) reads

$$\phi'(u^2)B(x^1) = 0$$

which gives the following two cases.

Case 1. $\phi(u^2)$ *arbitrary function.* The symmetry infinitesimal generator is

$$X = c_1\partial_{x^1} = c_1 X_1 \tag{40}$$

with c_1 arbitrary constant.

Case 2. $\phi(u^2) = \phi_0 =$ *constant.* The symmetry infinitesimal generator is

$$X = c_1\partial_{x^1} + B(x^1)\partial_{u^2} \tag{41}$$

with $B(x^1)$ arbitrary function. The symmetry Lie algebra is infinite-dimensional and it is spanned by

$$X_1 \qquad \text{and} \qquad X_B = B(x^1)\partial_{u^2}.$$

The other cases can be analysed in a similar way. We summarize the results of the symmetry classification obtained from the WETs in table 2.

Remark 1. The principal Lie algebra $L_{\mathcal{P}}$ of the symmetry group, that is the algebra of the symmetry group admitted for each functional form of the arbitrary functions $\alpha(u^1)$, $\gamma(x^2)$ and $\phi(u^2)$, is generated by X_1, the time translation.

Remark 2. If the search for symmetries is performed via CETs [2, 13] one obtains fewer symmetries than those obtained by using the WETs. In fact, as shown in [1], in the case γ = constant, by employing the CETs only the case B = constant is recovered and the corresponding symmetry Lie algebra becomes finite-dimensional.

5. A class of solutions in cases of physical interest

In this section we apply the results of the previous sections by considering a case of particular physical interest described by means of a system of PDEs of the type (1), (2): the DDM for the charge transport in semiconductors.

The DDM, obtained starting from the transport equation for electrons in a semiconductor crystal lattice by means of a Hilbert expansion, is represented by the balance equation for the charge density, coupled with the Poisson equation for the electric potential (see [17–19]).

In the unipolar version (only the motion of the electrons is considered while the motion of the holes is neglected) it reads

$$\partial_t n = \partial_x(\alpha\partial_x n - nv(E)) \tag{42}$$

$$\partial_x E = n - \gamma(x) \tag{43}$$

with n, v, E and $\gamma(x)$ representing the scaled electron number density, velocity, electric field and doping profile, respectively.

Usually equation (43) is substituted by the equation for the electric potential, but for one-dimensional problems the algebra is simplified by considering the equation for the electric field.

The diffusion coefficient α may depend on n and E. In this paper we assume that α depends only on n. The general case will be considered in a forthcoming paper.

The previous system falls into the class (1), (2) setting $x^1 = t$, $x^2 = x$, $u^1 = n$ and $u^2 = E$.

In order to close the system (42), (43), a relation between v and E must be assigned. This depends on the type of semiconductor and it is obtained on the basis of asymptotic expansions or fittings of experimental data or Monte Carlo simulations. In figure 1 we show the typical behaviour of the velocity versus the electric field for Si and GaAs.

Here we shall restrict ourselves to the case

$$\phi = \phi_1 E + \phi_0 \qquad \alpha = \text{constant} \qquad \gamma = \text{constant}.$$

A comparison with figure 1 shows that such a relation is a good approximation for a low electric field.

For this choice of ϕ, α and γ the symmetries of the resulting system (42), (43) are generated by X_1 and X_D. The case $D(x^1)$ = constant leads to invariant solutions of the form of travelling waves, which have been analysed to study the Gunn effects in GaAs semiconductors [18].

Here we shall consider the case $D(x^1) \neq 0$. Without loss of generality, the scaled doping is put equal to one and we denote the length of the device by l.

We look for solutions which are invariant with respect to

$$X = c_1\partial_t + D(t)\partial_x + \frac{D'(t)}{\phi_1}\partial_E.$$

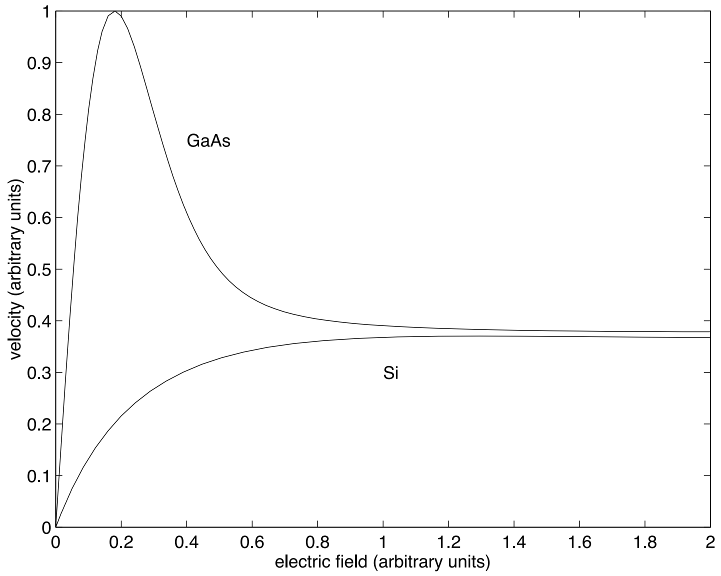

Figure 1. Drift velocity versus electric field in arbitrary units for silicon (Si) and gallium arsenide (GaAs).

From the characteristic equations

$$\frac{\mathrm{d}t}{c_1} = \frac{\mathrm{d}x}{D(t)} = \frac{\phi_1\,\mathrm{d}E}{D'(t)}$$

the invariant basis of the symmetry group is obtained:

$$I_1 = R(t) - c_1 x$$
$$I_2 = n$$
$$I_3 = E - \frac{R'(t)}{c_1\phi_1}$$

with $R(t) = \int D(t)\,\mathrm{d}t$.

Then the invariant solutions have the form

$$n = U(\sigma)$$
$$E = \frac{R'(t)}{c_1\phi_1} + V(\sigma).$$

where $\sigma := I_1$.

After substituting into the system (42), (43), we get the *reduced* system

$$\alpha c_1 U'' + \phi_1 U' V + \phi_0 U' + \phi_1 V' U = 0 \tag{44}$$
$$-c_1 V' = U - 1 \tag{45}$$

where primes denote derivatives with respect to σ.

Equation (44) has the first integral

$$\alpha c_1 U' + \phi_1 U V + \phi_0 U = k_0 = \text{constant}.$$

Then, by using equation (45), a single second-order differential equation is obtained:

$$\alpha c_1^2 V'' + (1 - c_1 V')(\phi_1 V + \phi_0) = k_0.$$

By introducing the transformation

$$W = \phi_1 V + \phi_0$$

and by setting $c_1 = 1$, one gets

$$\alpha W'' + W W' - \phi_1 (W + k_0) = 0. \tag{46}$$

For $k_0 = 0$, the general solution of (46) is given in an implicit form:

$$\sigma - \sigma_0 = \int_{n(\sigma_0)}^{n(\sigma)} \left[\phi_1 + \phi_1 L \left(\frac{1}{\phi_1} \exp(\frac{2\phi_1 - \tau^2 + 2c)}{2\phi_1} \right) \right]^{-1} \mathrm{d}\tau \tag{47}$$

where c is an arbitrary constant and L is the Lambert function, implicitly defined by

$$L(\zeta) \exp(L(\zeta)) = \zeta.$$

Note that the previous solution solves the system (42), (43) with the following boundary conditions depending on an arbitrary function of time:

$$n(0, t) = U(R(t)) \qquad n(l, t) = U(R(t) - l) \qquad E(0, t) = R(t) + V(R(t)).$$

6. Conclusions

In this paper Lie symmetries for a class of a drift–diffusion system have been found by following a different procedure from the direct Lie infinitesimal method. Even if this approach does not guarantee obtaining complete symmetry classification, it provides a systematic way for obtaining wide classes of symmetries when arbitrary functions appear and the Lie direct infinitesimal criterion becomes too involved to be successfully applied.

The followed method is based on the weak equivalence classification already introduced in [10–12]. Here we improve the results of those papers by generalizing the suggestion presented in [13, 14], where all the coordinates of the infinitesimal generators were sought depending also on the arbitrary functions. The corresponding prolongation formulae of any order are given in the appendix.

Finally, as an application of the results of the classification, a class of solutions in the case of charge transport in semiconductors is obtained. We stress that the functional form of the invariant base allows us to deal with boundary conditions containing an arbitrary function of time.

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Appendix. Prolongation formulae for equivalence transformations

Let us consider the transformation (7)–(9).

In order to get the prolongation formulae of the infinitesimal generators of arbitrary order

$$\Gamma^{(n)} = \Gamma^{(n-1)} + \zeta^{\alpha}_{i_1 \ldots i_n} \partial_{u^{\alpha}_{i_1 \ldots i_n}} + \omega^{A}_{a_1 \ldots a_n} \partial_{p^{A}_{a_1 \ldots a_{n+4}}} \tag{48}$$

related to such a transformation in the *n-order augmented jet space*

$$A^{(n)} = A \times A_1 \times \cdots \times A_n$$

we consider separately the components of Γ relative to the partial derivatives of the dependent functions u^α_j and the components of Γ relative to the partial derivatives of the arbitrary functions p^A_a.

Firstly, in order to determine the components $\zeta^\alpha_{i_1\ldots i_n}$, the variables u^α and p^A are sought as

$$u^\alpha = u^\alpha(x^j) \qquad \text{and} \qquad p^A = p^A(x^j, u^\beta(x^k)).$$

Therefore, from the transformation (7)–(9) it follows that

$$\overline{x}^j = \overline{x}^j(x^i).$$

If we denote the total derivative with respect to x^j by D^e_j, and the total derivative with respect to $\overline{x}^j$ by $\overline{D}^e_j$, from the transformation rule

$$D^e_j \overline{u}^\alpha = (D^e_j \overline{x}^k)(\overline{D}^e_k u^\alpha)$$

and the infinitesimal form of the transformations (7)–(9)

$$\overline{x}^k = x^k + \varepsilon \xi^k(x^i, u^\beta, p^A) + \mathrm{o}(\varepsilon)$$
$$\overline{u}^\alpha = u^\alpha + \varepsilon \eta^\alpha(x^i, u^\beta, p^A) + \mathrm{o}(\varepsilon)$$
$$\overline{u}^\alpha_j = u^\alpha_j + \varepsilon \zeta^\alpha_j(x^i, u^\beta, p^A) + \mathrm{o}(\varepsilon)$$

formula (12) is easly obtained.

More generally, by induction the recursive relations are proved by the standard procedure:

$$\zeta^\alpha_{i_1\ldots i_n} = D^e_{i_n} \zeta^\alpha_{i_1\ldots i_{n-1}} - u^\alpha_{i_1\ldots i_{n-1}k} D^e_{i_n} \xi^k. \tag{49}$$

Now, in order to find the components $\omega^A_a \partial_{p^A_a}$, we consider x^k and u^α as independent functions and the p^A are sought as

$$p^A = p^A(x^k, u^\beta).$$

Therefore, from the transformations (7)–(9)

$$\overline{z}^a = \overline{z}^a(z^b).$$

By considering the relation

$$\tilde{D}_a \overline{p}^A = (\tilde{D}^e_a \overline{z}^b) \overline{p}^A_{\overline{z}^b}$$

and by taking into account the infinitesimal form of the transformations for the partial derivatives of p^A,

$$\overline{p}^A_{a_1\ldots a_n} = p^A_{a_1\ldots a_n} + \varepsilon \omega^A_{a_1\ldots a_n} + \mathrm{o}(\varepsilon)$$

we get formula (13).

Proceeding by induction we suppose that

$$\omega^A_{a_1\ldots a_{n-1}} = \tilde{D}^e_{a_{n-1}}(\omega^A_{a_1\ldots a_{n-2}}) - p^A_{a_1\ldots a_{n-2}b} \tilde{D}^e_{a_{n-1}} \nu^b.$$

Then, from the relation

$$\tilde{D}^e_{a_n} \overline{p}^A_{a_1\ldots a_{n-1}} = (\tilde{D}^e_{a_n} \overline{z}^b) \overline{p}^A_{\overline{z}_{a_1}\ldots \overline{z}_{a_{n-1}} \overline{z}^b}$$

written for the corresponding infinitesimal transformation

$$\tilde{D}^e_{a_n}(p^A_{a_1\ldots a_{n-1}} + \varepsilon \omega^A_{a_1\ldots a_{n-1}} + \mathrm{o}(\varepsilon)) = (\tilde{D}^e_{a_n}(z^b + \varepsilon \nu^b + \mathrm{o}(\varepsilon)))(p^A_{a_1\ldots a_{n-1}b} + \varepsilon \omega^A_{a_1\ldots a_{n-1}b} + \mathrm{o}(\varepsilon))$$

it follows that

$$\omega^A_{a_1\ldots a_n} = \tilde{D}^e_{a_n}(\omega^A_{a_1\ldots a_{n-1}}) - p^A_{a_1\ldots a_{n-1}b} \tilde{D}^e_{a_n} \nu^b. \tag{50}$$

We summarize the previous results in the following theorem.

Theorem 2. *Let us consider the equivalence transformation (7)–(9). The prolongation of order n of* Γ *is given by (48) with* $\zeta^{\alpha}_{i_1\ldots i_n}$ *and* $\omega^{A}_{a_1\ldots a_n}$ *given by (49) and (50).*

The prolongation formulae obtained above trivially extend to transformations with an arbitrary number of dependent and independent variables and arbitrary functions.

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